GAINESVILLE STATE COLLEGE
University System of Georgia

Gainesville State College
Thirteenth Annual Mathematics Tournament
April 14, 2007

Morning Component

Good morning!

Please do NOT open this booklet until given the signal to begin.

There are 40 multiple choice questions. Answer the questions on the electronic grading form by giving the best answer to each question.

The scoring will be done by giving one point for each question answered correctly and zero points for each question answered incorrectly or left blank. Thus, it is to your advantage to answer as many questions as possible, even if you have to guess. If there is a tie, question number 18 will be used again as a tie-breaker.

This test was designed to be a CHALLENGE. It is difficult, and you may not have time to complete all questions. Do not worry if you are unable to answer several of the questions. Instead, we hope that you will obtain satisfaction from those questions which you ARE able to answer.

You may write in the test booklet. You may keep your test booklet and any of your scrap papers. Only the electronic grading form will be collected and graded.

Good luck!

Do Not Open Until Signaled.

If you need this document in another format, please email minsu.kim@ung.edu or call 678−717−3546.

Thirteenth Annual Gainesville State College Mathematics Tournament

You may write in this test booklet. Only the electronic form will be graded. Correct answers are awarded one point. Incorrect or blank answers are awarded 0 points.

1. Evaluate $\lim_{x\to\frac{\pi}{6}} \dfrac{\sin\left(x+\frac{\pi}{3}\right)-1}{x-\frac{\pi}{6}}$.

   a) 1

   b) -1

   c) 0

   d) $\frac{1}{2}$

   e) none of the above

2. Evaluate $\int_2^3 \dfrac{2x-1}{\left(4x^2-4x\right)^2}\,dx$.

   a) $\frac{1}{16}$

   b) $\frac{1}{32}$

   c) $\frac{1}{48}$

   d) $\frac{1}{64}$

   e) none of the above

3. Suppose f is differentiable on $(-\infty,\infty)$, $f(-1)=4$, and $|f'(x)|\le 1$ for all x. Which of the following could be true?

I: f is strictly increasing.
II: f is strictly decreasing.
III: $f(\sqrt{2})=\sqrt{2}$

a) I only
b) I and II
c) I and III
d) I, II, and III
e) none of the above

4. How many horizontal lines are tangent to the curve $(x^2+y^2)^2=2(x^2-y^2)$?

a) 0
b) 1
c) 2
d) 3
e) none of the above

5. Find $\lim_{x\to\infty}\left(\sqrt{x^2-x}-\sqrt{x^2+x}\right)$.

a) ∞
b) $-\infty$
c) 0
d) -1
e) none of the above

6. Suppose $f'(x)=\sec^2(x)$ and $f(0)=1$. What can be said about $f(\pi)$?

a) $f(\pi)=0$
b) $f(\pi)=1$
c) $f(\pi)$ is undefined.
d) There is not enough information to determine $f(\pi)$ uniquely.
e) none of the above

7. Evaluate $\int_0^{\infty} 3xe^{-4x}\,dx$.

a) $\frac{3}{4}$

b) $\frac{3}{8}$

c) $\frac{3}{16}$

d) $\frac{3}{32}$

e) none of the above

8. Write an equation for the tangent line to the graph of $f(x)=\sin(x)$ at $x=\frac{\pi}{4}$.

a) $y=\frac{\sqrt{2}}{2}x+\frac{\sqrt{2}(4-\pi)}{8}$

b) $y=-\frac{\sqrt{2}}{2}x+\frac{\sqrt{2}(4-\pi)}{8}$

c) $y=\frac{\sqrt{2}}{2}x-\frac{\sqrt{2}(4-\pi)}{8}$

d) $y=-\frac{\sqrt{2}}{2}x-\frac{\sqrt{2}(4-\pi)}{8}$

e) none of the above

9. Find the indefinite integral $\int \frac{dx}{x\sqrt{1-(\ln x)^2}}$.

a) $\ln(x)\sin^{-1}(x)+C$

b) $x\sin^{-1}(x)+C$

c) $\ln\left(\sin^{-1}(x)\right)+C$

d) $\sin^{-1}(\ln(x))+C$

e) none of the above

10. Let f and g be differentiable functions such that $f'(x)=g(x)$ and $g'(x)=-f(x)$, and let $T(x)=\left[f(x)\right]^2+\left[g(x)\right]^2$. Find $T'(x)$.

a) $T'(x)=4f(x)g(x)$

b) $T'(x)=2$

c) $T'(x)=2f(x)g(x)$

d) $T'(x)=0$

e) none of the above

11. If the slope of a strictly monotonic function f is $\frac{4}{9}$ at a particular point (a,b), what is the slope of f^{-1} at the point (b,a)?

a) $\frac{9}{4}$

b) -5

c) $-\frac{9}{4}$

d) 5

e) none of the above

12. The graph of a function f consists of a semicircle and two line segments as shown below. Let $g(x)=\int_0^x f(t)\,dt$. What is <u>not</u> true about the function g?

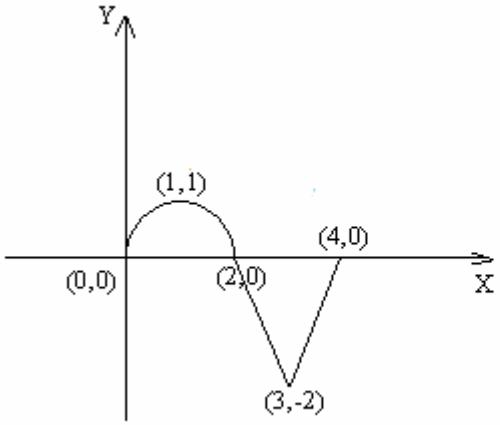
a) $g(1)=\frac{\pi}{4}$

b) $g(2)=\frac{\pi}{2}$

c) $g(3)$ is positive

d) $g(4)$ is positive

e) none of the above



13. A tank filled with water is in the shape of an inverted cone 20 *ft* high with a circular base (on top) whose radius is 5 *ft*. Water is running out of the bottom of the cone at the constant rate of 2 ft^3 / min. How fast is the level of water falling when the water is 8 *ft* deep?

a) $\frac{1}{\pi}$ *ft/min*

b) $\frac{1}{2\pi}$ *ft/min*

c) $\frac{1}{3\pi}$ *ft/min*

d) $\frac{1}{5\pi}$ *ft/min*

e) none of the above

14. Let $h(x) = f(g(x))$ where $f'(x) = \sin x$, $g'(x) = 4x$, and $g(3) = 15$. Find $h'(x)$.

a) $4x\sin\left(2x^2 - 3\right)$

b) $4x\cos\left(2x^2 - 3\right)$

c) $4x\sin\left(2x^2 + 3\right)$

d) $4x\cos\left(2x^2 + 3\right)$

e) none of the above

15. An apartment building has 200 apartments. When the building is fully occupied, the rent for each apartment is $750 per month. However, the owner knows from past experience, for each additional $25 increase in monthly rent, 2 additional apartments will become vacant. How much should the owner charge for rent to maximize his revenue?

a) $1125
b) $1625
c) $973
d) $1100
e) none of the above

16. Determine which of the following is <u>not</u> equal to the definite integral $\int_2^7 axf(x)\,dx$.

a) $a\int_2^7 xf(x)\,dx$

b) $x\int_2^7 af(x)\,dx$

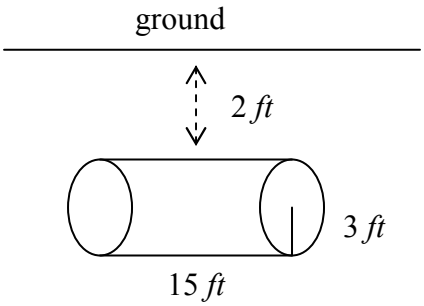
c) $-\int_7^2 axf(x)\,dx$

d) $\int_2^4 axf(x)\,dx - \int_7^4 axf(x)\,dx$

e) none of the above

17. A cylindrical gas tank with radius 3 *ft* and length 15 *ft* is buried 2 *ft* below ground level. The density of gasoline is $\rho = 46\ lb/ft^3$. Find the amount of work required to pump all of the gas out of a full tank.

a) 31,050 *ft-lbs*
b) 6,210 *ft-lbs*
c) $6{,}210\pi$ *ft-lbs*
d) $31{,}050\pi$ *ft-lbs*
e) none of the above



Reminder
Question 18 will be used again as a tie-breaker, if necessary.

18. What values of *a* and *b* with $a < b$, maximize the value of $\int_a^b \left(3x - x^2\right)dx$?

a) $a = -\infty,\ b = 0$
b) $a = 0,\ b = \infty$
c) $a = 0,\ b = 3$
d) $a = -3,\ b = 0$
e) none of the above

19. Find the length of the curve $y = x^{2/3}$, $1 \le x \le 8$.

a) $\frac{8}{27}\left(80\sqrt{10} - 13\sqrt{13}\right)$

b) $\frac{1}{27}\left(80\sqrt{10} - 13\sqrt{13}\right)$

c) $\frac{8}{27}\left(13\sqrt{13} - 8\sqrt{10}\right)$

d) $\frac{1}{27}\left(8\sqrt{10} - 13\sqrt{13}\right)$

e) none of the above

20. Let $f(x)$ be a continuous function over the interval $(-\infty, \infty)$. Given $\int_a^b f(x)\,dx = 3$, $\int_c^d f(x)\,dx = -2$, $\int_b^d f(x)\,dx = 5$, where *a*, *b*, *c*, and *d* are real numbers with $a < b < c < d$. What is $\int_a^c f(x)\,dx$?

a) 12
b) 10
c) 8
d) 6
e) none of the above

21. Find the number of discontinuity points of the function

$$f(x) = \begin{cases} x+1 & \text{if } x \le 0 \\ \frac{1}{2} & \text{if } 0 < x \le 1 \\ 2-x & \text{if } 1 < x \le 2 \\ (2-x)^2 & \text{if } 2 < x \end{cases}$$

a) 0
b) 1
c) 2
d) 3
e) none of the above

22. Find the area of the region bounded by $y=\sqrt{9-x^2}$, $y=\sqrt{4-x^2}$ and $y=0$.

a) 5π
b) $\frac{5\pi}{2}$
c) $\frac{5\pi}{4}$
d) $\frac{5\pi}{9}$
e) none of the above

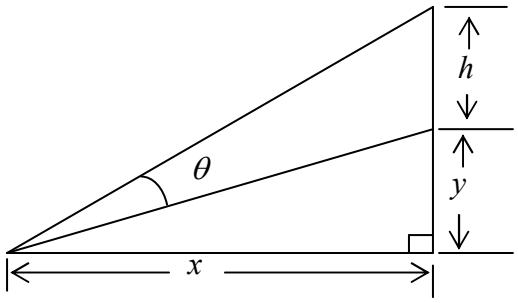
23. Find the average value of $f(x)=\begin{cases}\frac{x^2}{\sin x} & 0<|x|\le\frac{\pi}{2}\\ 0 & x=0\end{cases}$ over $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.

a) 1
b) -1
c) $\frac{1}{\pi}$
d) $\frac{\pi}{4}$
e) none of the above

24. A driver of a car applies the brakes and experiences a constant acceleration of $-42\ ft/sec^2$ and the car skids for d feet before coming to a stop. The velocity of the car, in ft/sec, at the time the driver applies the brakes is given by

a) $\frac{\sqrt{7d}}{21}$
b) $\frac{\sqrt{21d}}{21}$
c) $2\sqrt{7d}$
d) $2\sqrt{21d}$
e) none of the above

25. What value of x maximizes θ in the given figure:



a) $\sqrt{y^2+y}$

b) $\sqrt{y^2+yh}$

c) $\sqrt{y^2+h}$

d) $\sqrt{y^2h+yh^2}$

e) none of the above

26. Suppose $f(0)=0,\ f(1)=1,\ f'(x)>0$ for all x, and $f''(x)>0$ for all x. Then

a) $\int_0^1 f(x)\,dx < \int_0^1 f^{-1}(y)\,dy$

b) $\int_0^1 f(x)\,dx = \int_0^1 f^{-1}(y)\,dy$

c) $\int_0^1 f(x)\,dx > \int_0^1 f^{-1}(y)\,dy$

d) $f^{-1}(y)$ might not exist

e) none of the above

27. Consider $F(x)=\int_x^1 \sqrt{1+t^2}\,dt$. Find $F'(x)$.

a) $\sqrt{1+x^2}$

b) $\dfrac{x}{\sqrt{1+x^2}}$

c) $\dfrac{1}{2\sqrt{1+x^2}}$

d) $-\sqrt{1+x^2}$

e) none of the above

28. If $x=0$ is a critical number of the function f, then what can be said about the function $g(x)=f(x-h)+k$, where h and k are any real numbers?

a) $x=-h$ is a critical number of g
b) $x=0$ is a critical number of g
c) $x=k$ is a critical number of g
d) $x=h$ is a critical number of g
e) none of the above

29. Identify which integral computes the volume of the solid formed by revolving the region bounded by the graphs of $y=\ln(x)$, the x-axis, and the line $x=3$ about the line $x=3$.

a) $\pi\int_1^3\left[9-(\ln x)^2\right]dx$
b) $\pi\int_0^{\ln 3}\left[9-e^{2y}\right]dy$
c) $\pi\int_0^{\ln 3}\left[3-e^{y}\right]^2 dy$
d) $\pi\int_1^3\left[\ln x\right]^2 dx$
e) none of the above

30. Consider $F(x)=\int_{\cos x}^{\sin x}\sqrt{1-t^2}\,dt$ where x is in the interval $\left[0,\frac{\pi}{2}\right]$. Find $F'(x)$.

a) $F'(x)=\cos(2x)$
b) $F'(x)=\sin(2x)$
c) $F'(x)=\cos(x)-\sin(x)$
d) $F'(x)=1$
e) none of the above

31. Find the volume of the solid of revolution formed by revolving about the *x*-axis the region between the line $y = 0$ and $f(x) = e^{\frac{-x}{2}}\sqrt{\cos(x)+1}$, with $x \geq 0$.

a) $\frac{3}{2}\pi$

b) π

c) 3π

d) $\frac{1}{2}\pi$

e) none of the above

32. If f is twice differentiable and $\int_0^x f''(t)\,dt > 0$ for all x in $(0,1]$, then

a) f must be concave up on $[0,1]$

b) f must be increasing on $[0,1]$

c) f must be decreasing on $[0,1]$

d) f must be positive on $[0,1]$

e) none of the above

33. If f is continuous on $[-2,3]$ with $f(-2) = 5$ and $f(3) = -4$, which of the following must be true?

I: $f'(x) = -\frac{9}{5}$ has a solution with $-2 < x < 3$

II: $-4 \leq f(x) \leq 5$ if $-2 < x < 3$

III: f attains a maximum value on $[-2,3]$

a) I only
b) II only
c) III only
d) I, II, III
e) none of the above

34. Suppose $f(x) = x^5 + 2x^3 + 7x - 4$ and f^{-1} denotes the inverse of f.

Then $\left(f^{-1}\right)'(6) =$

a) $\frac{1}{21}$

b) $\frac{1}{20}$

c) $\frac{1}{18}$

d) $\frac{1}{12}$

e) none of the above

35. Water is flowing into a tank at the rate of $\sqrt{3t+1}$ ft^3 / min, where t is measured in minutes. If the tank is empty at $t = 0$, how much water does it contain 5 minutes later?

a) 16 ft^3

b) 14 ft^3

c) 12 ft^3

d) 10 ft^3

e) none of the above

36. For a differentiable function f, find $\lim_{h \to 0} \frac{f(x+h) - f(x-h)}{h}$.

a) $2f'(x)$

b) $f'(x)$

c) $\frac{1}{2} f'(x)$

d) $f(x)$

e) none of the above

37. The volume of a right circular cone is $V = \frac{2\sqrt{2}}{3}\pi$. Find the smallest possible *total* surface area of such a cone.

a) 4π
b) $2\pi\left(1+\sqrt{2}\right)$
c) 3π
d) $2\pi\sqrt{2}$
e) none of the above

38. Given $y = x^3 + bx^2 + 3x + 5$, what condition must be imposed on b so that the function has no relative maxima or minima?

a) $|b| \le 3$
b) $|b| \ge 3$
c) $b > 3$
d) $b \le -3$
e) none of the above

39. Let g be a differentiable function of x. If $f(x) = \frac{g(x)}{x^2}$ for $x > 0$, $g(2) = 3$, and $g'(2) = -2$, then $f'(2) =$

a) $\frac{5}{4}$
b) $-\frac{5}{4}$
c) $-\frac{1}{2}$
d) 5
e) none of the above

40. Find the area of the region bounded by the curves $f(x) = x + 1$ and $g(x) = x^2 - 2x + 1$.

a) $\frac{9}{2}$

b) 9

c) $\frac{3}{4}$

d) $\frac{5}{2}$

e) none of the above