Gainesville State College

Fourteenth Annual Mathematics Tournament

April 12, 2008

Solutions for the Afternoon Team Competition

Round 1

Let x be the number of rabbits, y be the number of kittens, and z the number of chickens. We have

$$(1)\ x+y+z=100$$
$$(2)\ 2x+y+0.1z=100$$

The third condition gives us $z=\frac{2}{3}(x+y)$ or $3z=2x+2y$.
Multiply equation (1) by 2 to obtain $2x+2y+2z=200 \Rightarrow 3z+2z=200 \Rightarrow z=40$.
Substituting back and reducing, we have

$$x+y=60 \qquad \text{and} \qquad 2x+y=96$$

This has solutions $x=36$ and $y=24$. So, the number of chickens is 40, the number of rabbits is 36, and the number of kittens is 24.

Round 2

Let A and C represent the area and circumference, respectively. Then,

$$A=6C+60 \Rightarrow \pi r^2=6\cdot 2\pi r+60 \quad \Rightarrow \quad \pi r^2-2\cdot 6\pi r-60=0 \quad \Rightarrow \quad 2\left(\frac{\pi}{2}r^2-6\pi r-30\right)=0$$

$$\text{So } r=\frac{-(-6\pi)+\sqrt{(-6\pi)^2-4\left(\frac{\pi}{2}\right)(-30)}}{2\left(\frac{\pi}{2}\right)}=\frac{6\pi+\sqrt{36\pi^2+60\pi}}{\pi}\approx 13.423$$

Round 3

To get a zero at the end of a number, you need to multiply a 2 and a 5 together. There are fewer factors of 5 in numbers between 1 and 100 than there are factors of 2. So the number of factors of 5 contained in 100! determines the number of zeros at the end.

$$100! = (\text{the other integers without factors of } 5)\left(100\cdot 95\cdot 90\cdot \ldots \cdot 5\right).$$

The tables show all the integers with factors of 5 that are in 100!

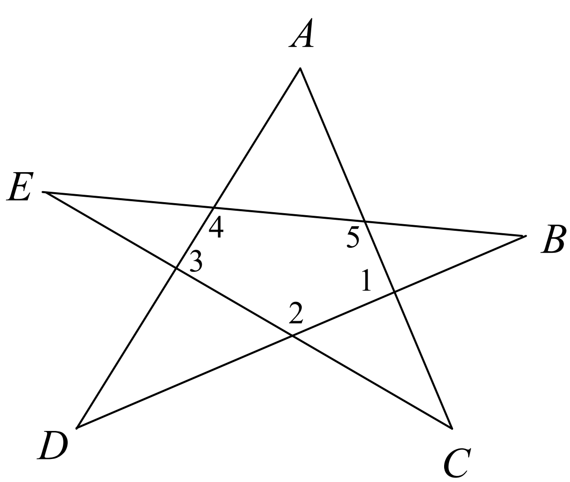
Integers
$5=1\cdot 5$
$10=2\cdot 5$
$15=3\cdot 5$
$20=4\cdot 5$
$25=1\cdot 5^2$
$30=6\cdot 5$
$35=7\cdot 5$
$40=8\cdot 5$
$45=9\cdot 5$
$50=2\cdot 5^2$

Integers
$55=11\cdot 5$
$60=12\cdot 5$
$65=13\cdot 5$
$70=14\cdot 5$
$75=3\cdot 5^2$
$80=16\cdot 5$
$85=17\cdot 5$
$90=18\cdot 5$
$95=19\cdot 5$
$100=4\cdot 5^2$

There are a total of 24 factors of 5.

Therefore, there are 24 zeros at the end of 100!

Round 4



$$\angle A+\angle 1+\angle D=180^\circ$$

$$\angle B+\angle 2+\angle E=180^\circ$$

$$\angle C+\angle 3+\angle A=180^\circ$$

$$\angle D+\angle 4+\angle B=180^\circ$$

$$\angle E+\angle 5+\angle C=180^\circ$$

$$2\left(\angle A+\angle B+\angle C+\angle D+\angle E\right)+3\cdot 180^\circ = 5\cdot 180^\circ$$

Therefore, $\angle A+\angle B+\angle C+\angle D+\angle E=180^\circ$

Round 5

The discriminant is $b^2 - 4c$. If $b^2 - 4c < 0$, then $b^2 < 4c$ and the quadratic equation has no real solutions. Consider the following:

1. When $b = 1$, $b^2 = 1 \Rightarrow 1 < 4c \Rightarrow c > \frac{1}{4}$. So $c = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$.
   Thus, 10 such equations.

2. When $b = 2$, $b^2 = 4 \Rightarrow 4 < 4c \Rightarrow c > 1$. So $c = 2, 3, 4, 5, 6, 7, 8, 9, 10$.
   Thus, 9 such equations.

3. When $b = 3$, $b^2 = 9 \Rightarrow 9 < 4c \Rightarrow c > \frac{9}{4}$. So $c = 3, 4, 5, 6, 7, 8, 9, 10$.
   Thus, 8 such equations.

4. When $b = 4$, $b^2 = 16 \Rightarrow 16 < 4c \Rightarrow c > 4$. So $c = 5, 6, 7, 8, 9, 10$.
   Thus, 6 such equations.

5. When $b = 5$, $b^2 = 25 \Rightarrow 25 < 4c \Rightarrow c > \frac{25}{4}$. So $c = 7, 8, 9, 10$.
   Thus, 4 such equations.

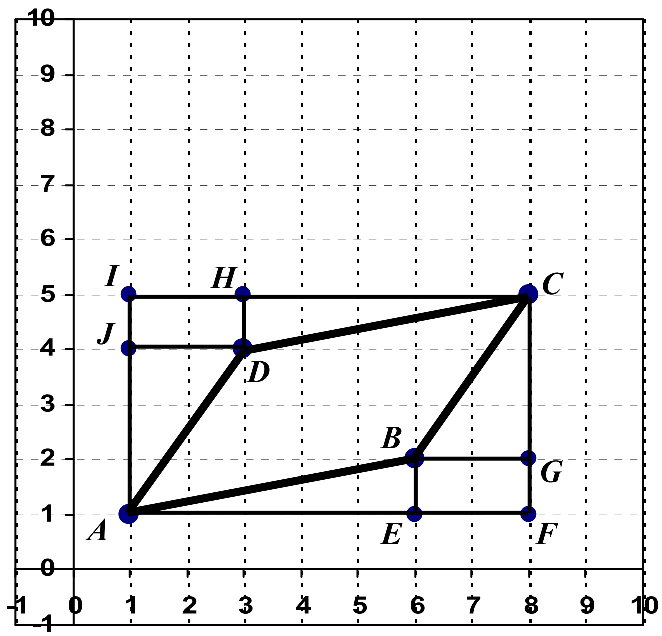
6. When $b = 6$, $b^2 = 36 \Rightarrow 36 < 4c \Rightarrow c > 9$. So $c = 10$.
   Thus, 1 such equation.

7. When $b = 7$, $b^2 = 49 \Rightarrow 49 < 4c \Rightarrow c > \frac{49}{4} > 10$. So there are no more considerations.

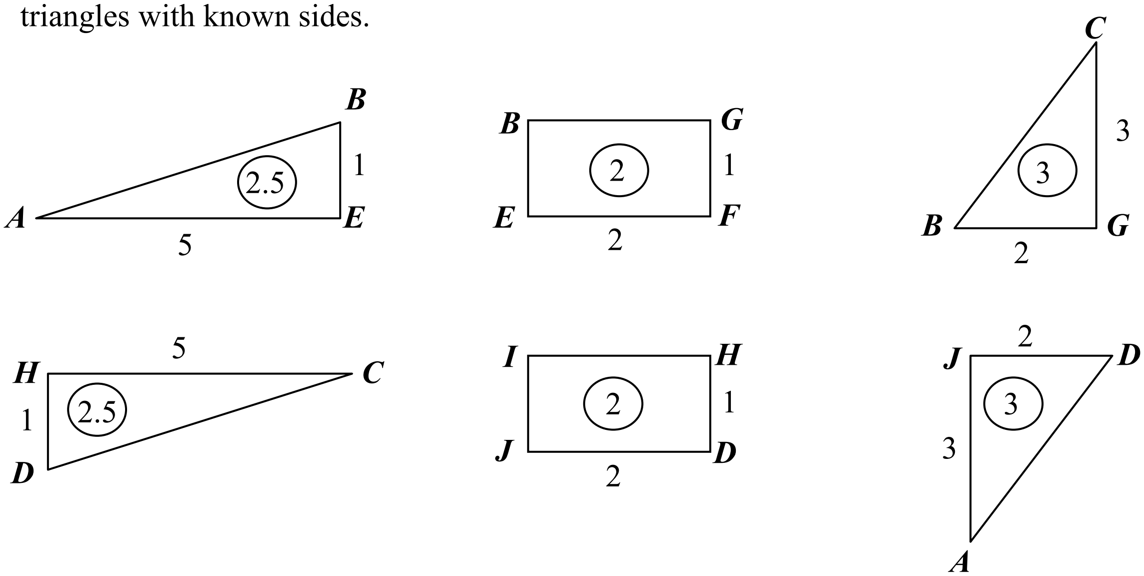
Hence, there are a total of $10 + 9 + 8 + 6 + 4 + 1 = 38$ such equations.

Round 6

Let's introduce points E, F, G, H, I, and J as shown in the picture.



Then the areas outside of the parallelogram can be obtained as areas of rectangles or right triangles with known sides.



Then from the area of the rectangle *AFCI* (equal to $4 \cdot 7 = 28$), we subtract the areas of the outside triangles and rectangles, obtaining the area of the parallelogram.

$$28 - (2.5 + 2 + 3 + 2.5 + 2 + 3) = 13$$

Round 7

$$\left(\frac{1\cdot 2\cdot 4+2\cdot 4\cdot 8+3\cdot 6\cdot 12+\ldots}{1\cdot 3\cdot 9+2\cdot 6\cdot 18+3\cdot 9\cdot 27+\ldots}\right)^{\frac{1}{3}}=\left[\frac{1\cdot 2\cdot 4\left(1^3+2^3+3^3+\ldots\right)}{1\cdot 3\cdot 9\left(1^3+2^3+3^3+\ldots\right)}\right]^{\frac{1}{3}}=\left(\frac{8}{27}\right)^{\frac{1}{3}}=\frac{2}{3}$$

Round 8

Let x = Mary's current age and y = John's current age. We get these equations:

(1) $x-1=2(y-1)$

(2) $x+10=5y$

Solving the system gives $x=5$ and $y=3$. Therefore, Mary is 5 years old.

Round 9

We know $P(1)=3$ and $P(3)=5$. Let $D(x)=(x-1)(x-3)=x^2-4x+3$. If $P(x)$ is divided by $D(x)$, we can write $P(x)=D(x)\cdot Q(x)+R(x)$ where $Q(x)$ is the quotient and $R(x)$ is the remainder. Since $D(x)$ is a second degree polynomial, the general form for $R(x)$ is $ax+b$. Thus, $P(x)=D(x)\cdot Q(x)+ax+b$. Then

$P(1)=D(1)\cdot Q(1)+a\cdot 1+b$ and we know $P(1)=3$ and $D(1)=0$. Therefore, $3=a+b$.

$P(3)=D(3)\cdot Q(3)+a\cdot 3+b$ and we know $P(3)=5$ and $D(3)=0$. Therefore, $5=3a+b$.

Thus, $a=1$ and $b=2$. So the remainder is $x+2$.

Round 10

$$\log_2 x+\log_4 x+\log_8 x=\log_2 x+\frac{\log_2 x}{\log_2 4}+\frac{\log_2 x}{\log_2 8}\Rightarrow \log_2 x+\frac{\log_2 x}{2}+\frac{\log_2 x}{3}=11$$

Multiplying by 6, we have $6\log_2 x+3\log_2 x+2\log_2 x=66\Rightarrow 11\log_2 x=66\Rightarrow \log_2 x=6$

Therefore, $x=2^6$ or $x=64$.