University of North Georgia

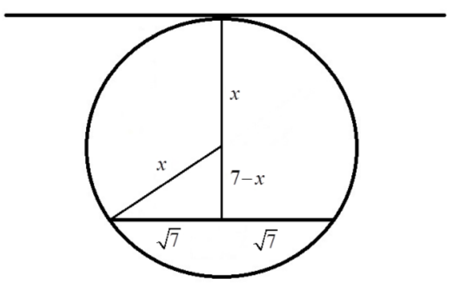
Mathematics Tournament

April 1, 2017

Solutions for the Afternoon Team Competition

Round 1

From the figure below: $\left(\sqrt{7}\right)^2+\left(7-x\right)^2=x^2$. Simplifying gives $7+49-14x+x^2=x^2$ and then $56=14x$ and $x=4$.



Round 2

All *x*, *y*, *z* are nonnegative real numbers. Using change of base, rewrite the system as:

$$\frac{\log x}{\log 2}+\frac{\log y}{2\log 2}+\frac{\log z}{2\log 2}=2$$
$$\frac{\log y}{\log 3}+\frac{\log z}{2\log 3}+\frac{\log x}{2\log 3}=2$$
$$\frac{\log z}{\log 4}+\frac{\log x}{2\log 4}+\frac{\log y}{2\log 4}=2$$

Thus, getting common denominators and clearing fractions gives:

$$2\log x + \log y + \log z = 4\log 2 \qquad (1)$$
$$2\log y + \log z + \log x = 4\log 3 \qquad (2)$$
$$2\log z + \log x + \log y = 4\log 4 \qquad (3)$$

Adding all three equations we get: $4\log x + 4\log y + 4\log z = 4(\log 2 + \log 3 + \log 4)$.

This simplifies to: $4\log(xyz) = 4\log(2\cdot 3\cdot 4)$ and then $x^4y^4z^4 = 2^4 3^4 4^4$ (4).

Equation (3) – (1) gives: $\log z - \log x = 4\log 4 - 4\log 2$. Hence, $\frac{z}{x} = 2^4$ and $\frac{z}{2^4} = x$.

Equation (3) – (2) gives: $\log z - \log y = 4\log 4 - 4\log 3$. Hence, $\frac{z}{y} = \frac{4^4}{3^4}$ and $\frac{3^4 z}{4^4} = y$.

Substituting into (4), we get: $\left(\frac{z^4}{2^{16}}\right)\left(\frac{3^{16}z^4}{4^{16}}\right)(z^4) = 2^4 3^4 4^4$. Hence, $(z^{12})(3^{16})(2^{-48}) = (2^{12})(3^4)$.

Thus, $z^{12} = 2^{60}3^{-12}$. Hence, $z = \frac{2^5}{3} = \frac{32}{3}$. Hence, $x = \frac{\frac{32}{3}}{2^4} = \frac{2}{3}$ and $y = \frac{3^4 \cdot \frac{32}{3}}{4^4} = \frac{3^3}{2^3} = \frac{27}{8}$.

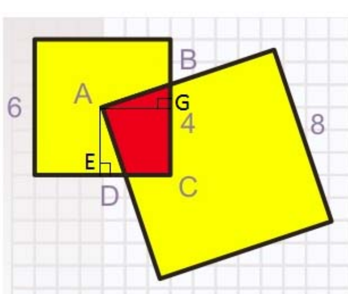
This gives the ordered triplet $\left(\frac{2}{3}, \frac{27}{8}, \frac{32}{3}\right)$.

Round 3

For $x = 1$, $f(1) + 2f(5) = 1$. For $x = 5$, $f(5) + 2f(1) = 5$. Solving for $f(5)$ in the second equation gives $f(5) = 5 - 2f(1)$. Substituting into the first equation gives $f(1) + 2(5 - 2f(1)) = 1$. Solving gives $f(1) = 3$.

Round 4

Triangle *AED* and triangle *AGB* are congruent because $\angle AED = \angle AGB = 90°$, $\overline{AE} = \overline{AG} = 3$, and $\angle EAD = \angle GAB$, because they both equal $90° - \angle DAG$. So they have the same area. Thus the area of *ABCD* (shaded region) is the same as the area of *AECG* which is one quarter of the area of the small square $\left(\frac{1}{4}(6\cdot 6) = \frac{1}{4}(36) = 9\right)$. The area is 9 square units.



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Round 5

The sum of interior angles of a regular *n*-sided polygon is $180°(n-2)$. So $180(n-2)=1800$. Simplifying gives $n-2=10$ and $n=12$.

The area of a regular *n*-sided polygon with side length *s* is $A=\dfrac{s^2n}{4\tan\left(\dfrac{\pi}{n}\right)}=\dfrac{1}{4}s^2n\cot\left(\dfrac{\pi}{n}\right)$.

In this case $s=\dfrac{1}{\sqrt{3}}$ and $n=12$, so $A=\dfrac{1}{4}\left(\dfrac{1}{\sqrt{3}}\right)^2(12)\cot\left(\dfrac{\pi}{12}\right)=\dfrac{1}{4}\cdot\dfrac{1}{3}\cdot 12\cot\left(\dfrac{\pi}{12}\right)$. Simplifying further gives:

$$A=\cot\left(\frac{\pi}{12}\right)=\cot\left(\frac{\pi}{3}-\frac{\pi}{4}\right)=\frac{1+\tan\frac{\pi}{3}\tan\frac{\pi}{4}}{\tan\frac{\pi}{3}-\tan\frac{\pi}{4}}=\frac{1+\sqrt{3}\cdot 1}{\sqrt{3}-1}=\frac{1+\sqrt{3}}{\sqrt{3}-1}\cdot\frac{\sqrt{3}+1}{\sqrt{3}+1}=\frac{\sqrt{3}+1+3+\sqrt{3}}{3-1}=\frac{4+2\sqrt{3}}{2}=2+\sqrt{3}$$

Round 6

If more than 24 people show up, there will not be enough seats since there are only 24 seats. So $x>24$ or $x=25$ and $x\sim Bin(25,0.92)$.

Hence, $P(x=25)=\binom{25}{25}(0.92)^{25}(1-0.92)^{25-25}=(1)(0.92)^{25}(0.08)^{0}\approx 0.1244=12.44\%$

Round 7

From the first equation: $x^4y^4(x+y)=810$. From the second equation: $x^3y^3(x^3+y^3)=945$.

Now $2x^3+(xy)^3+2y^3=2(x^3+y^3)+(xy)^3=2\left(\dfrac{945}{x^3y^3}\right)+x^3y^3$.

Cubing both sides of the first equation gives:

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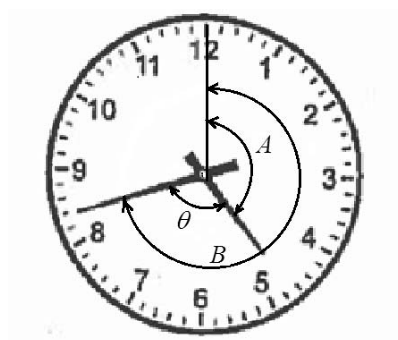
$$\left[x^4y^4(x+y)\right]^3=(810)^3$$
$$x^{12}y^{12}\left(x^3+y^3+3x^2y+3xy^2\right)=810^3$$
$$x^{12}y^{12}\left(x^3+y^3+3xy(x+y)\right)=810^3$$
$$x^{12}y^{12}\left(x^3+y^3+3xy\cdot\frac{810}{x^4y^4}\right)=810^3$$
$$x^{12}y^{12}\left(\frac{945}{x^3y^3}+3\cdot\frac{810}{x^3y^3}\right)=810^3$$
$$x^9y^9(945+3\cdot 810)=810^3$$
$$x^3y^3=\frac{810}{15}=54$$

So (from above) $2x^3+(xy)^3+2y^3=2\left(\frac{945}{x^3y^3}\right)+x^3y^3=\frac{1890}{54}+54=89$.

Round 8



From the picture, $\theta=B-A$, where *B* and *A* are the measures in degrees of the minute hand and hour hand with respect to the 12 o'clock position.

To find *B*, note that $1'=\frac{360}{60}=6°$, hence the minute hand creates (with 12 o'clock) angle $B=42'=6\cdot 42=252°$.

To find *A*, note that $1\,hour=\frac{360}{12}=30°$. The hour hand rotates $30°$ in one hour (60 minutes) and $0.5°$ in one minute. Since *A* corresponds to the time 4:42 or 4 hours and 42 minutes, $A=(4\cdot 30°)+(42\cdot 0.5°)=120°+21°=141°$.

Thus, $\theta=B-A=252°-141°=111°$.

Round 9

The sides increase by 20%, so each side increases by a factor of 1.2. The height of the triangle will increase by the same factor. Thus the area would increase by a factor of $(1.2)^2=1.44$ which is an increase of 44%.

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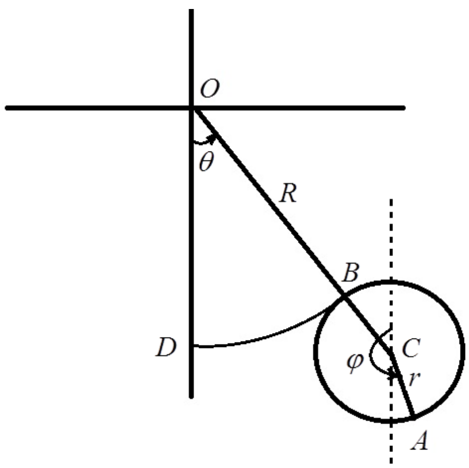
Round 10

The angle θ locates the center *C* of the smaller cylinder with respect to a fixed vertical reference. The point *A*, which is on the rolling cylinder, and point *D*, which is on the stationary cylinder, are coincident when $\theta = 0$. The distance $DB = BA$ where these are arc lengths on the larger and smaller cylinders respectively.

So $R\theta = r(\phi - \theta)$. Simplifying gives $(R + r)\theta = r\phi$ and $\phi = \left(\frac{R}{r} + 1\right)\theta$. When $R = 3$ and $r = 1$, we have $\phi = \left(\frac{3}{1} + 1\right)\theta = 4\theta = 4(1\ rotation) = 4\ rotations$.



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